Stochastic Optimization

An Introduction

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The Farmer's Problem

Consider a European farmer who specializes in raising grain, corn, and sugar beets on his 500 acres of land. During the winter, he wants to decide how much land to devote to each crop.
The farmer knows that at least 200 tons (T) of wheat and 240 T of corn are needed for cattle feed. These amounts can be raised on the farm or bought from a wholesaler. Any production in excess of the feeding requirement would be sold. Selling prices are $\$170$ and $\$150$ per ton of wheat and corn, respectively. The purchase prices are 40% more than this due to the wholesaler's margin and transportation costs.
Another profitable crop is sugar beet, which sells at $\$36/$T; however, the European Commission imposes a quota on sugar beet production. Any amount in excess of the quota can be sold only at $\$10/$T. The farmer's quota for next year is 6000 T.

The Farmer's Problem

TABLE 1. Data for farmer's problem.

	Wheat	Corn	Sugar Beets
Yield (T/acre)	2.5	3	20
Planting cost ($/acre)	150	230	260
Selling price ($/T)	170	150	36 under 6000 T 10 above 6000 T
Purchase price ($/T)	238	210	–
Minimum requirement (T)	200	240	–
Total available land: 500 acres			

The Farmer's Problem

x_1 = acres of land devoted to wheat,
x_2 = acres of land devoted to corn,
x_3 = acres of land devoted to sugar beets,
w_1 = tons of wheat sold,
y_1 = tons of wheat purchased,
w_2 = tons of corn sold,
y_2 = tons of corn purchased,
w_3 = tons of sugar beets sold at the favorable price,
w_4 = tons of sugar beets sold at the lower price.

$$
\begin{aligned}
\min \quad & 150x_1 + 230x_2 + 260x_3 + 238y_1 - 170w_1 \\
& +210y_2 - 150w_2 - 36w_3 - 10w_4 \\
\text{s.t. } & x_1 + x_2 + x_3 \leq 500,\ 2.5\ x_1 + y_1 - w_1 \geq 200, \\
& 3\ x_2 + y_2 - w_2 \geq 240, w_3 + w_4 \leq 20x_3, w_3 \leq 6000, \\
& x_1, x_2, x_3, y_1, y_2, w_1, w_2, w_3, w_4 \geq 0.
\end{aligned}
$$

The Farmer's Problem

The farmer becomes worried. He has indeed experienced quite different yields for the same crop over different years mainly because of changing weather conditions. Most crops need rain during the few weeks after seeding or planting, then sunshine is welcome for the rest of the growing period. Sunshine should, however, not turn into drought, which causes severe yield reductions. Dry weather is again beneficial during harvest. From all these factors, yields varying 20 to 25% above or below the mean yield are not unusual.

To fix these ideas, "above" and "below" average indicate a yield 20% above or below the mean yield given in Table 1.

The Farmer's Problem

TABLE 3. Optimal solution based on above average yields (+ 20%).

Culture	Wheat	Corn	Sugar Beets
Surface (acres)	183.33	66.67	250
Yield (T)	550	240	6000
Sales (T)	350	–	6000
Purchase (T)	–	–	–
Overall profit: $167,667			

The Farmer's Problem

TABLE 4. Optimal solution based on below average yields (−20%).

Culture	Wheat	Corn	Sugar Beets
Surface (acres)	100	25	375
Yield (T)	200	60	6000
Sales (T)	–	–	6000
Purchase (T)	–	180	–
Overall profit: \$59,950			

The Farmer's Problem

The farmer now realizes that he is unable to make a perfect decision that would be best in all circumstances. He would, therefore, want to assess the benefits and losses of each decision in each situation. Decisions on land assignment (x_1, x_2, x_3) **have to be taken now**, but sales and purchases $(w_i, i = 1, \ldots, 4, y_j, j = 1, 2)$ depend on the yields.
It is useful to index those decisions by a scenario index $s = 1, 2, 3$ corresponding to above average, average, or below average yields, respectively.

The Farmer's Problem

$$
\begin{aligned}
\min \quad 150x_1 + 230x_2 + 260x_3 & - \frac{1}{3}(170w_{11} - 238y_{11} + 150w_{21} \\
& -210y_{21} + 36w_{31} + 10w_{41}) \\
& - \frac{1}{3}(170w_{12} - 238y_{12} + 150w_{22} \\
& -210y_{22} + 36w_{32} + 10w_{42}) \\
& - \frac{1}{3}(170w_{13} - 238y_{13} + 150w_{23} \\
& -210y_{23} + 36w_{33} + 10w_{43})
\end{aligned}
$$

$$
\begin{gathered}
\text{s.t.} \\
x_1 + x_2 + x_3 \le 500,\ 3x_1 + y_{11} - w_{11} \ge 200, 3.6x_2 + y_{21} - w_{21} \ge 240, \\
w_{31} + w_{41} \le 24x_3,\ w_{31} \le 6000,\ 2.5x_1 + y_{12} - w_{12} \ge 200, \\
3x_2 + y_{22} - w_{22} \ge 240,\ w_{32} + w_{42} \le 20x_3,\ w_{32} \le 6000, \\
2x_1 + y_{13} - w_{13} \ge 200,\ 2.4x_2 + y_{23} - w_{23} \ge 240,\ w_{33} + w_{43} \le 16x_3, \\
w_{33} \le 6000,\ \ x, y, w \ge 0.
\end{gathered}
$$

Decisions and Stages

Stochastic Linear Programs

...are linear programs in which some problem data may be considered uncertain.

Recourse programs

...are those in which some decisions or recourse actions can be taken after uncertainty is disclosed.

The set of decisions is then divided into two groups

- A number of decisions have to be taken before the experiment. All these decisions are called **first-stage decisions** and the period when these decisions are taken is called the **first stage**.
- A number of decisions can be taken after the experiment. They are called **second-stage decisions**. The corresponding period is called the **second stage**.

Two-Stage Program with Fixed Recourse

$$
\begin{aligned}
\min z = & c^T x + E_{\boldsymbol{\xi}}[\min q(\omega)^T y(\omega)] \\
& \text{s.t. } Ax = b, \\
& T(\omega)x + W y(\omega) = h(\omega), \\
& x \geq 0 \ , y(\omega) \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
\min z = & c^T x + \mathcal{Q}(x) \\
& \text{s.t. } Ax = b, \\
& x \geq 0,
\end{aligned}
$$

$$\mathcal{Q}(x) = E_{\boldsymbol{\xi}} Q(x, \xi(\omega))$$

$$Q(x, \xi(\omega)) = \min_y \{q(\omega)^T y | W y = h(\omega) - T(\omega)x, y \geq 0\}.$$

Examples of recourse formulation and interpretations

The Location Problem.

Let $i = 1, \ldots, m$ index clients having demand d_i for a given commodity. The firm can open a facility (such as a plant or a warehouse) in potential sites $j = 1, \ldots, n$. Each client can be supplied from an open facility where the commodity is made available (i.e., produced or stored). The problem of the firm is to choose the number of facilities to open, their locations, and market areas to maximize profit or minimize costs.

The Location Problem. Deterministic version

Decision Variables

$$x_j = \begin{cases} 1 & \text{if facility } j \text{ is open} \\ 0 & \text{otherwise} \end{cases}$$

y_{ij} = fraction of demand of client i served from facility j

Cost & Revenue Parameters

c_j = fixed cost to open/operate facility j
v_j = variable operating cost of facility j
t_{ij} = unit transportation cost from j to i
r_i = unit price charged to client i
d_i = demand of client i

Total Revenue

If all of client i's demand is satisfied from facility j:

$$q_{ij} = (r_i - v_j - t_{ij})d_i$$

where q_{ij} is the total revenue obtained.

The Location Problem. Determinist version

$$
\begin{aligned}
\max_{x,y}\ z(x,y) =& -\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{ij} y_{ij} \\
\text{s.t.} & \sum_{j=1}^{n} y_{ij} \le 1, i = 1, \cdots, m, \\
& 0 \le y_{ij} \le x_j, i = 1, \ldots, m, j = 1, \ldots, n, \\
& x_j \in \{0,1\}, j = 1, \ldots, n.
\end{aligned}
$$

The Location Problem

Several parameters of the problem may be uncertain and may thus have to be represented by random variables. Production and distribution costs may vary over time. Future demands for the product may be uncertain.

It is customary to consider that the location decisions x_j are first-stage decisions because it takes some time to implement decisions such as moving or building a plant or warehouse. The main modeling issue is on the distribution decisions. The firm may have full control on the distribution, for example, when the clients are shops owned by the firm. It may then choose the distribution pattern after conducting some random experiments. In other cases, the firm may have contracts that fix which plants serve which clients, or the firm may wish fixed distribution patterns in view of improved efficiency because drivers would have better knowledge of the regions traveled.

Fixed distribution pattern, fixed demand, r_i , v_j , t_{ij} stochastic

Assume the only uncertainties are in production and distribution costs and prices charged to the client.

We now show that the problem is in fact a deterministic problem in which the total revenue $q_{ih} = (r_i - v_j - t_{ij})d_i$ can be replaced by its expectation.

Fixed distribution pattern, fixed demand, r_i , v_j , t_{ij} stochastic

$$\max \quad -\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \sum_{j=1}^{n} (E_{\boldsymbol{\xi}} q_{ij}(\omega)) y_{ij}$$

Fixed distribution pattern, uncertain demand

Fixed distribution pattern, uncertain demand

$$
\begin{aligned}
\max \quad & -\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m}\sum_{j=1}^{n} (E_{\boldsymbol{\xi}}(-v_j - t_{ij})) y_{ij} + E_{\boldsymbol{\xi}}[-\sum_{i=1}^{m} q_i^+ w_i^+(\omega) \\
& \qquad - \sum_{i=1}^{m} q_i^- w_i^-(\omega)] + E_{\boldsymbol{\xi}} \sum_{i=1}^{m} r_i d_i \\
\text{s.t.} \quad & \sum_{i=1}^{m} y_{ij} \le M x_j, \ j = 1, \ldots, n, \\
& w_i^+(\omega) - w_i^-(\omega) = d_i(\omega) - \sum_{j=1}^{n} y_{ij}, i = 1, \ldots, m, \\
& x_j \in \{0,1\}, 0 \le y_{ij}, w_i^+(\omega) \ge 0, w_i^-(\omega) \ge 0, i = 1, \ldots, m; j = 1, \ldots, n.
\end{aligned}
$$

Uncertain demand, variable distribution pattern

Uncertain demand, variable distribution pattern

$$\begin{aligned}
\max \quad & -\sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{n} g_j w_j + E_{\boldsymbol{\xi}} \max \sum_{i=1}^{m} \sum_{j=1}^{n} q_{ij}(\omega) y_{ij}(\omega) \\
\text{s.t.} \quad & x_j \in \{0,1\},\ w_j \geq 0, j = 1, \ldots, n, \\
& \sum_{j=1}^{n} y_{ij}(\omega) \leq 1, i = 1, \ldots, m, \\
& \sum_{i=1}^{m} d_i(\omega) y_{ij}(\omega) \leq w_j, j = 1, \ldots, n, \\
& 0 \leq y_{ij}(\omega) \leq x_j, i = 1, \ldots, m, j = 1, \ldots, n,
\end{aligned}$$

Stages versus periods; Two-stage versus multistage

Stages versus periods; Two-stage versus multistage

$$
\begin{aligned}
\max \quad & -\sum_{j=1}^{n} c_j x_j^1 + E_{\boldsymbol{\xi}_2} \max\{\sum_{i=1}^{m}\sum_{j=1}^{n} q_{ij}^2(\omega_2) y_{ij}^2(\omega_2) - \sum_{j=1}^{n} c_j^2(\omega_2) x_j^2(\omega_2) \\
& + E_{\boldsymbol{\xi}_3|\boldsymbol{\xi}_2} \max[\sum_{i=1}^{m}\sum_{j=1}^{n} q_{ij}^3(\omega_3) y_{ij}^3(\omega_3)]\} \\
\text{s.t.} \quad & \sum_{j=1}^{n} y_{ij}^2(\omega_2) \leq 1, i = 1, \ldots, m, \\
& \sum_{i=1}^{m} d_i(\omega_2) y_{ij}^2(\omega_2) \leq M x_j^1, j = 1, \ldots, n, \\
& \sum_{j=1}^{n} y_{ij}^3(\omega_3) \leq 1, i = 1, \ldots, m, \\
& \sum_{i=1}^{m} d_i(\omega_3) y_{ij}^3(\omega_3) \leq M(x_j^1 + x_j^2(\omega_2)), j = 1, \ldots, n, \\
& x_j^1 + x_j^2(\omega_2) \leq 1, j = 1, \ldots, n, \\
& x_j^1, x_j^2(\omega_2) \in \{0, 1\}, j = 1, \ldots, n, \\
& y_{ij}^2(\omega_2), y_{ij}^3(\omega_3) \geq 0, i = 1, \ldots, m, j = 1, \ldots, n.
\end{aligned}
$$

Two-Stage Stochastic Linear Programs with Fixed Recourse

$$
\begin{aligned}
\min z = & c^T x + E_{\boldsymbol{\xi}}[\min q(\omega)^T y(\omega)] \\
& \text{s.t. } Ax = b, \\
& T(\omega)x + W y(\omega) = h(\omega), \\
& x \geq 0 \ , y(\omega) \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
\min z = & c^T x + \mathcal{Q}(x) \\
& \text{s.t. } Ax = b, \\
& x \geq 0,
\end{aligned}
$$

$$\mathcal{Q}(x) = E_{\boldsymbol{\xi}} Q(x, \xi(\omega))$$

$$Q(x, \xi(\omega)) = \min_y \{q(\omega)^T y | W y = h(\omega) - T(\omega)x, y \geq 0\}.$$

Feasibility Sets

For fixed x and ξ, the value $\mathcal{Q}(x,\xi)$ of the second-stage program is given by

$$Q(x,\boldsymbol{\xi}) = \min_y \{q(\omega)^T y | W(\omega)y = h(\omega) - T(\omega)x, y \geq 0\}.$$

When the above mathematical expression is unbounded below or infeasible, the value of the second-stage program is defined to be $-\infty$ or $+\infty$, respectively.

We let defined
$K_1 = \{x | Ax = b, x \geq 0\}$, $K_2 = \{x | \mathcal{Q}(x) < \infty\}$.
We may now redefine the deterministic equivalent program as follows

$$\begin{aligned} \min \quad & z(x) = c^T x + \mathcal{Q}(x) \\ & \text{s.t. } x \in K_1 \cap K_2. \end{aligned}$$

Feasibility Sets

$$
\begin{aligned}
K_2^P &= \{x |\ \text{for all } \xi \in \Xi, \\
&\quad y \geq 0 \text{ exists s.t. } Wy = h - T \cdot x\} \\
&= \cap_{\xi \in \Xi} K_2(\xi).
\end{aligned}
$$

The set K_2^P is said to define the possibility interpretation of second-stage feasibility sets.

Theorem 1.
a. For each ξ, the elementary feasibility set is a closed convex polyhedron, hence the set K_2^P is closed and convex.
b. When Ξ is finite, then K_2^P is also polyhedral and coincides with K_2.

Feasibility Sets

Proposition 2. *If* $\boldsymbol{\xi}$ *has finite second moments, then*

$$P(\omega | Q(x, \boldsymbol{\xi}) < \infty) = 1 \quad \text{implies } \mathcal{Q}(x) < \infty.$$

Feasibility Sets

Theorem 3. *For a stochastic program with fixed recourse where* $\boldsymbol{\xi}$ *has finite second moments, the sets* K_2 *and* K_2^P *coincide.*

Feasibility Sets

Theorem 4. *When W is fixed and $\boldsymbol{\xi}$ has finite second moments:*

(a) K_2 *is closed and convex.*

(b) If T *is fixed,* K_2 *is polyhedral.*

(c) Let Ξ_T *be the support of the distribution of* $\mathbf{T}$*. If* $h(\boldsymbol{\xi})$ *and* $T(\boldsymbol{\xi})$ *are independent and* Ξ_T *is polyhedral, then* K_2 *is polyhedral.*

Feasibility Sets

Theorem 5. *For a stochastic program with fixed recourse,* $Q(x, \xi)$ *is*

(a) a piecewise linear convex function in (h, T)*;*

(b) a piecewise linear concave function in q*;*

(c) a piecewise linear convex function in x *for all* x *in* $K = K_1 \cap K_2$*.*

Optimality Conditions

Theorem 9. Suppose that our problem has a finite optimal value. A solution $x^\star \in K_1$ is optimal if and only if there exists some $\lambda^\star$ and $\mu^\star$, $\langle \mu^\star, x^\star \rangle = 0$, such that,

$$-c + A^T \lambda^\star + \mu^\star \in \partial \mathcal{Q}(x^\star).$$

The Expected Value of Perfect Information

(EVPI) measures the maximum amount a decision maker would be ready to pay in return for complete (and accurate) information about the future.

Suppose uncertainty can be modeled through a number of scenarios. Let ξ be the random variable whose realizations correspond to the various scenarios.
Define

$$\begin{aligned} \min z(x,\xi) &= c^T x + \min\{q^T y \mid Wy = h - Tx, y \geq 0\} \\ \text{s.t.} \quad Ax &= b, x \geq 0, \end{aligned}$$

The Expected Value of Perfect Information

Let $\bar{x}(\xi)$ denote some optimal solution to the problem. As in a scenario approach, we might be interested in finding all solutions $\bar{x}(\xi)$ of the problem for all scenarios and the related optimal objective values $z(\bar{x}(\xi), \xi)$.

The Expected Value of Perfect Information

Here, we assume we somehow have the ability to find these decisions $\bar{x}(\xi)$ and their objective values $z(\bar{x}(\xi), \xi)$ so that we are in a position to compute the expected value of the optimal solution, known in the literature as the *wait-and-see* solution (WS).

$$\begin{aligned} WS &= E_{\boldsymbol{\xi}}\left[\min_x z(x, \boldsymbol{\xi})\right] \\ &= E_{\boldsymbol{\xi}} z(\bar{x}(\boldsymbol{\xi}), \boldsymbol{\xi}). \end{aligned}$$

We may now compare the wait-and-see solution to the so-called *here-and-now* solution, (RP)

$$RP = \min_x E_{\boldsymbol{\xi}} z(x, \boldsymbol{\xi})$$

The expected value of perfect information is, by definition, the difference between the wait-and-see and the here-and-now solution, namely,

$$EVPI = RP - WS$$

The Expected Value of Perfect Information

An example was given in the farmer's problem. The wait-and-see solution value was $-\$115,406$ (when converted to a minimization problem), while the recourse solution value was $-\$108,390$. The expected value of perfect information for the farmer was then $\$7016$. This is how much the farmer would be ready to pay each year to obtain perfect information on next summer's weather. A meteorologist could reasonably ask him to pay part of this amount to support meteorological research.

The Value of the Stochastic Solution

For practical purposes, many people would believe that finding the wait-and-see solution or equivalently solving the distribution problem is still too much work (or impossible if perfect information is just not available at any price). This is especially difficult because the wait-and-see approach delivers a set of solutions instead of one solution that would be implementable.

A natural temptation is to solve a much simpler problem: the one obtained by replacing all random variables by their expected values. This is called the *expected value problem or mean value problem*, which is simply

$$EV = \min_{x} z(x, \bar{\xi})$$

The Value of the Stochastic Solution

We define the expected result of using the EV solution to be

$$EEV = E_{\boldsymbol{\xi}}(z(\bar{x}(\bar{\xi}), \boldsymbol{\xi}))$$

The quantity, EEV, measures how $\bar{x}(\xi)$ performs, allowing second-stage decisions to be chosen optimally as functions of $\bar{x}(\xi)$ and ξ. The value of the stochastic solution is then defined as

$$VSS = EEV - RP.$$

Recall for the farmer's problem, EEV $= -\$107, 240$ and RP $= -\$108, 390$, for V SS $= \$1150$. This quantity is the cost of ignoring uncertainty in choosing a decision.

Basic Inequalities

Proposition 1.

$$WS \leq RP \leq EEV$$

Propostion 2. For stochastic programs with fixed objective coefficients, fixed T, and fixed W,

$$EV \leq WS$$

Obrigado!

Figure: Illustration by Walter Molino